Quiz 3 – Solutions

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1. Define (for now) a *linear function* $f : \mathbb{R} \to \mathbb{R}$ to be a function for which $f(t) = mt$ for some constant m. *Using only this definition*, prove that for any linear function $f : \mathbb{R} \to \mathbb{R}$ and any real constants a, b, we have $f(ax + by) = af(x) + bf(y)$ for all $x, y \in \mathbb{R}$.

> **Solution:** By the given definition there exists $m \in \mathbb{R}$ such that $f(t) = mt$ for all $t \in \mathbb{R}$. Then, for any $x, y, a, b \in \mathbb{R}$,
>
> $$f(ax + by) = m(ax + by) = (ma)x + (mb)y = a(mx) + b(my) = a\, f(x) + b\, f(y).$$
>
> This uses only the distributive and associative properties of real numbers together with $f(t) = mt$.

2. True or False. If you answer true, then state TRUE. If you answer false, then state FALSE and provide a counterexample.

   (a) The solution set in $\mathbb{R}^4$ to three nontrivial linear equations in four unknowns can never be a line.

   > **Solution:** FALSE. Three independent, consistent linear equations in four variables can have rank 3, so the solution space has dimension $4 - 3 = 1$, i.e., a line.
   > **Counterexample:**
   >
   > $$\begin{cases} x_1 = 0, \\ x_2 = 0, \\ x_3 = 0. \end{cases} \qquad \text{(all nontrivial)}$$
   >
   > The solution set is $\{(0, 0, 0, t) : t \in \mathbb{R}\}$, which is a line in $\mathbb{R}^4$.

   (b) Suppose $m, n \in \mathbb{N}$, $\alpha \in \mathbb{R}$, and A, B are $n \times m$ matrices. Then
   - $A + B$ is an $n \times m$ matrix, and
   - αB is an $n \times m$ matrix.

   > **Solution:** TRUE. Matrix addition and scalar multiplication are defined entrywise. If $A = [a_{ij}]$ and $B = [b_{ij}]$ are both $n \times m$, then $A + B = [a_{ij} + b_{ij}]$ is also $n \times m$, and $\alpha B = [\alpha b_{ij}]$ is $n \times m$.